Counting processes and recurrent events beyond the cox model for Poisson process

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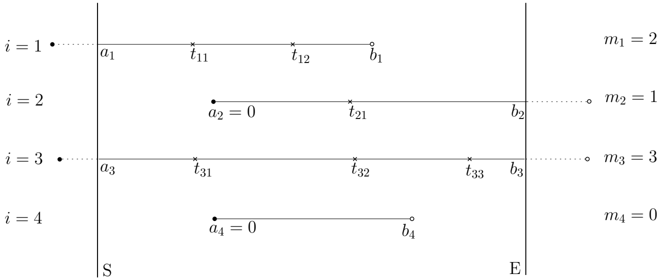
Dynamic predictions for repeated markers and repeated events
Workshop - GS0 2013

Outline

a joint work with
Karim Claudio (PhD LYRE, Suez), Genia Babykina (Post Doc,CRAN), Yves Legat (IRSTEA)

Counting processes and recurrent events

Type of data


$i = 1$
a_1
t_{11}
t_{12}
b_1
$m_1 = 2$
$i = 2$
$a_2 = 0$
t_{21}
b_2
$m_2 = 1$
$i = 3$
a_3
t_{31}
t_{32}
t_{33}
b_3
$m_3 = 3$
$i = 4$
$a_4 = 0$
b_4
$m_4 = 0$
S
E

Main Aim of analysis of recurrent events

- statistical analysis (and modeling) of non-independant occurence times of an event.
  - more than one event per individual.
  - interest in understanding the dependancy between times
- heterogenity and risk factors (covariates observed on individuals)
  - identify the covariates that influence the probability of event.
  - individual prediction of probability of occurrence knowing the characteristics of the individual
- Heterogeneity and frailties
  - does the intensity of events differ from individual to individual because of covariates or past history ?
  - dynamic intensity modeling versus frailty intensity modeling.

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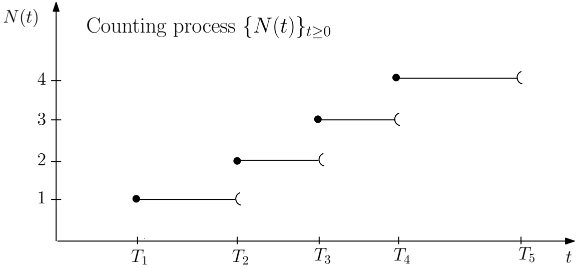
The counting process for recurrent events



Definition of the (stochastic) intensity

with respect to a history $\mathcal{H}(t)_{t>0}$(filtration generated by the known history) :

$$\begin{aligned} \lambda(t) &= \lim_{dt\to 0} \frac{1}{dt}\mathbb{P}\left[N(t+dt)-N(t)=1\,|\,\mathcal{H}(t-)\right] \\ \lambda(t)dt &= \mathbb{E}\left[dN(t)\,|\,\mathcal{H}(t-)\right], \end{aligned}$$

The marginal intensity : the rate function

- $\mathcal{H}(t)$ contains jump times and "external covariates" $Z(t)$,
  $\mathcal{H}(t) = \sigma\left(N(t), T_1, ..., T_{N(t)}, Z(t)\right)$

Note : $\lambda(t)$ is a predictable process, it captures the nature of the recurrence of events.

The rate function (ROCOF in reliability analysis)

$$
\begin{aligned}
r(t) &= \lim_{dt \to 0} \frac{1}{dt} \mathbb{P}\left[N(t+dt) - N(t) = 1 \,|\, Z(t-)\right] \\
r(t)dt &= \mathbb{E}\left[dN(t) \,|\, \mathcal{Z}(t-)\right],
\end{aligned}
$$

Ex : (Chiang, 1968)

$$
\begin{aligned}
\lambda(t) &= \beta_0(t) + \beta_1(t)Z + \beta_2(t)N(t-) \\
r(t) &= \beta_0(t) + \phi(\beta_1, \beta_2)Z + \psi(\beta_0, \beta_2)\beta_2(t),
\end{aligned}
$$

Recall the Cox model for survival analysis

- One event per subject → Survival analysis : $\lambda(t) = h(t)I_{N(t)=0}$

$$h(t) = \lim_{dt \to 0} \frac{1}{dt} P\left(T \in [t, t+dt[| T > t\right)$$

- regression model for covariate → Multiplicative intensity model

$$\lambda(t) = \lambda_0(t) e^{\beta_0 + \beta_1 Z_1 + \cdots + \beta_p Z_p} 1_{N(t-)=0}$$

Remark :

- A dead individual is no more "at risk"
- censoring mechanism may be considered **too** → "At risk" process $Y(t)$.

from Cox regression for lifetimes to Andersen and Gill intensity of counting processes

Remove The ”at risk” indicator $Y(t)$ to remains ”at risk” after the event.

Andersen & Gill, 82

$$\lambda(t)dt = \mathbb{E}\left[dN(t) \,|\, Z_1, \ldots Z_p\right] = \lambda_0(t)e^{\beta_0+\beta_1 Z_1+\cdots+\beta_p Z_p}\, dt$$

- The points of $N(t)_{t\geq 0}$ form a Poisson process conditionally on the values $Z_1, \ldots Z_p$.
- the process intensity does no depend on the past $\rightarrow$ not *dynamic*.

Models for the intensity process in reliability analysis

Some well known models for the intensity process

- dynamic intensity : impact of occurrence of an event on the intensity
  N(t-) or event times themselves ?
- observed "inter-unit" heterogenity : impact of covariates
  multiplicative intensity with "Cox-like" contribution.
- unobserved "inter-unit" heterogeneity : frailty : ? ?
  it may be hard to handle both frailty and dynamic parts.

Rk : in reliability (in this talk) :

- an event = failure + instantaneous maintenance/repair
- the dynamic part explains the maintenance actions.

The Poisson process with covariates

Cox regression model for recurrent events :

$$\lambda(t) = \lambda_0(t)e^{\beta' Z}, \text{ The Z's may be time-dependent}$$

- λ_0 usually increasing, ($\Rightarrow$ ageing).
- $e^{Z'\beta}$: impact of environment and/or individual heterogeneity

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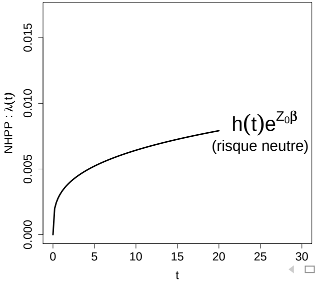

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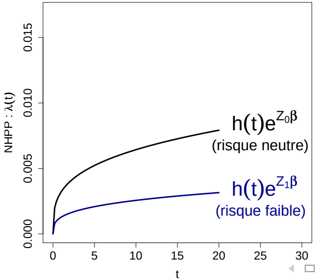

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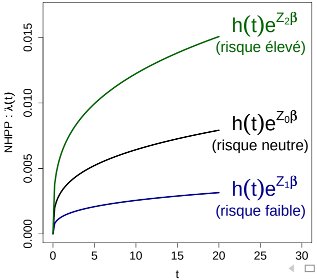
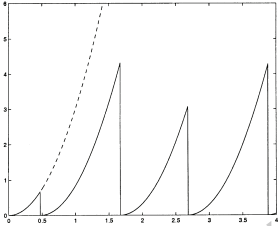


A renewal process for recurrent events

RP process (perfect repairs, AGAN models) :

$$\lambda(t) = \lambda_0(t - T_{N(t)})e^{\beta' Z},$$

- λ_0 usually increasing, $\lambda_0(0) = 0$ ($\Rightarrow$ ageing).
- the intensity uses the *duration* since the last event $t - T_{N(t)}$.
- inter-arrival durations are i.i.d. random variables
- *dynamic* intensity.



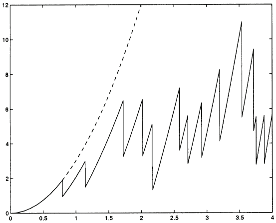
A generalized renewal process for recurrent events

GRP process (imperfect repair, virtual age models) :

$$\lambda(t) = \left(\lambda_0(t) - \rho\lambda_0\left(T_{N(t-)}\right)\right) e^{\beta' Z} \text{ ARI1}$$

$$\lambda(t) = \left(\lambda_0\left(t - \rho T_{N(t-)}\right)\right) e^{\beta' Z} \text{ ARA1}$$

- λ_0 usually increasing, $\lambda_0(0) = 0$ ($\Rightarrow$ ageing).
- ρ captures the *dynamic* of the reduction of intensity

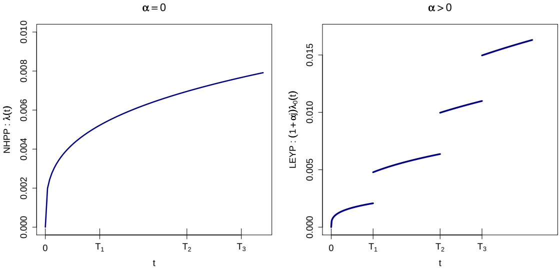


The linear extended Yule process for recurrent events

LEYP process (imperfect repair) :

$$\lambda(t) = (1 + \alpha N(t-))\lambda_0(t)e^{Z(t)'\beta}$$

- a dynamical component ($\alpha > 0$) uses the **number** of previous events.
- a baseline intensity λ_0 (usually parameterized, $\lambda_0(.,\theta)$).
- a Cox-like regression part $e^{Z(t)'\beta}$.



Generalization to Pena & Hollander model

The Pena-Hollander model

$$\lambda(t) = U\rho(N(t-), \alpha)\lambda_0(\epsilon(t))\psi\left(e^{Z(t)'\beta}\right)$$

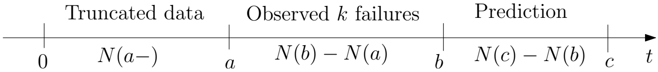
- a frailty component : U (non observable source of heterogeneity).
- a dynamical component $\rho(N(t-), \alpha)$
- a baseline intensity λ_0 (usually parameterized, $\lambda_0(., \theta)$).
- a Cox-like regression part $e^{Z(t)'\beta}$.

Remark : LEYP $\subset$ Pena & Hollander

Some useful properties for dynamic prediction and statistical estimation for the LEYP model.

Marginal and conditional distributions of N(t)

- $\lambda(t) = (1 + \alpha N(t-))\lambda_0(t; Z(t); \delta, \beta)$
- $\Lambda_0(t) = \int_0^t \lambda_0(s; Z(s); \delta, \beta) ds$



$$
\begin{aligned}
N(t) &\sim \mathcal{NB}(\alpha^{-1}, e^{-\alpha\Lambda_0(t)}) \\
\mathbb{E}[N(t)] &= \frac{e^{\alpha\Lambda_0(t)} - 1}{\alpha} \\
\mathbb{V}ar[N(t)] &= \frac{e^{\alpha\Lambda_0(t)}(e^{\alpha\Lambda_0(t)} - 1)}{\alpha}
\end{aligned}
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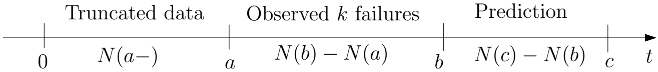

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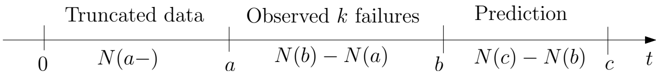


$$[N(b)-N(a)\,|\,N(a-)=k, Z(s), a<s<b] \sim \mathcal{NB}\left(\alpha^{-1}+k, e^{-\alpha[\Lambda_0(b)-\Lambda_0(a)]}\right)$$

$$[N(c)-N(b)\,|\,N(b-)-N(a)=k] \sim \mathcal{NB}\left(\alpha^{-1}+k, \frac{e^{\alpha\Lambda_0(b)}-e^{\alpha\Lambda_0(a)}+1}{e^{\alpha\Lambda_0(c)}-e^{\alpha\Lambda_0(a)}+1}\right)$$

The likelihood

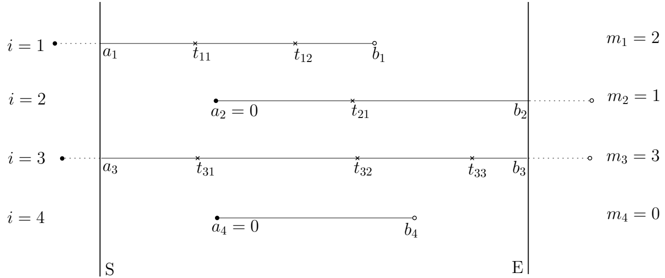
- One individual, observed on $[0, T]$, m events at times t_j ($j = \{1, \ldots m\}$) :

$$L(\theta) = \left(\prod_{j=1}^{m} \lambda(t_j)\right) \times \exp\left(-\sum_{j=0}^{m} \int_{t_j}^{t_{(j+1)}} \lambda(u)\, du\right)$$

Remark : In the following, parametric assumption on the baseline + truncated observation

- $\lambda(t) = (1 + \alpha N(t-))\delta t^{\delta-1} e^{Z(t)'\beta} = (1 + \alpha N(t-))\lambda_0(t; \delta, \beta)$
- N individuals, observed on $[a_i, b_i]$, $i = 1 \ldots N$
- $\Lambda_0(t) = \int_0^t \lambda_0(s) ds$

Data for maximum likelihood estimation



And : $Z_1, \ldots, Z_p$, fixed or external time-dependent covariates (no frailty)

The likelihood

- Log-likelihood for truncated data, N individuals

$$\begin{aligned}\ln L(\theta) = \sum_{i=1}^{N} \Big(& m_i \ln \alpha + \ln \Gamma(\alpha^{-1} + m_i) - \ln \Gamma(\alpha^{-1}) \\ - & (\alpha^{-1} + m_i) \ln(e^{\alpha \Lambda_0(b_i;\delta,\beta)} - e^{\alpha \Lambda_0(a_i;\delta,\beta)} + 1) \\ + & \sum_{j=1}^{m_i} (\ln \lambda_0(t_j;\delta,\beta) + \alpha \Lambda_0(t_j;\delta,\beta)) \Big)\end{aligned}$$

Appl. on recurrent failures of water networks

recurrent events of failure-repair on water network pipes

- SEDIF : A public drinking water service in area of Paris.
- stratification : only grey cast iron pipes are considered.
- 21450 pipes to provide drinking water (899km linear).
- failures recorded on 1996-2006 (11 years).
- mean age at inclusion : 35.8 years.
- mean duration of observation : 10 years.
- $\%$ of units with ≥ 1 failure : 12%.
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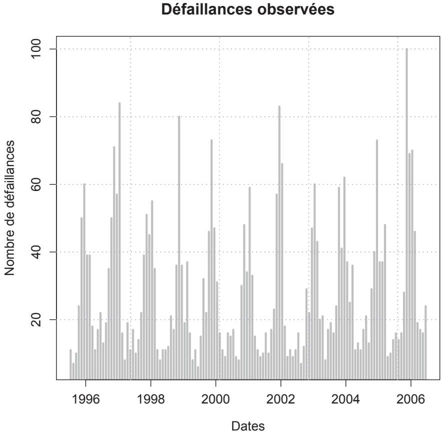
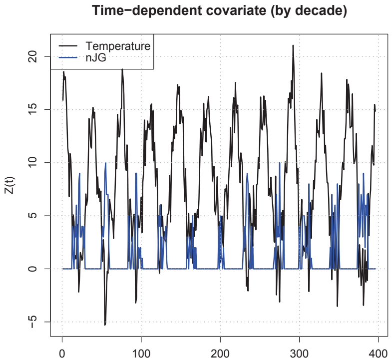


FIGURE : observed monthly number of failures

recurrent events of failure-repair on water network pipes

- the length L of the pipe influences the intensity
- the climate influences the intensity (time dependent)
  $\rightarrow X_2$: air temperature average over 10-days periods.

$$\lambda(t) = (1 + \alpha N(t-))\delta t^{\delta-1} e^{\beta_0+\beta_1 \ln(L)+\beta_2 X_2(t)}$$



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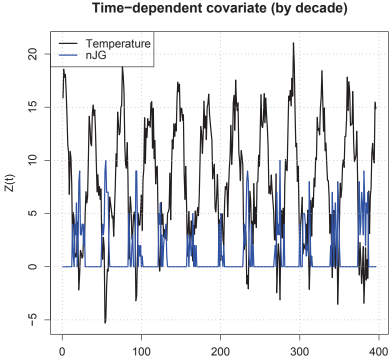

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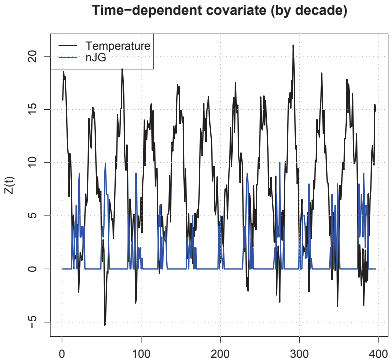

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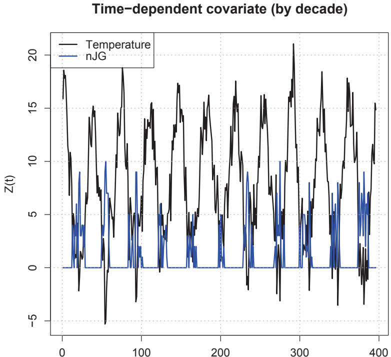


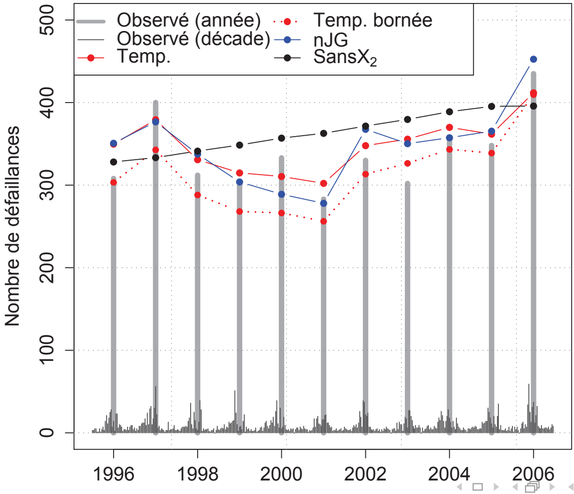
recurrent events of failure-repair on water network pipes

The results (maximum likelihood estimation)

Parameter	Model	Estimate	Standard Dev.	95% CI Estimate ± 1.96 std
α	with X_2	0.96	0.078	$[0.81, 1.12]$
	without X_2	0.98	0.079	$[0.82, 1.13]$
δ	with X_2	1.14	0.094	$[0.96, 1.32]$
	without X_2	1.11	0.094	$[0.93, 1.30]$
β_0	with X_2	-7.03	0.45	$[-7.92, -6.14]$
	without X_2	-7.52	0.45	$[-8.41, -6.63]$
β_1	with X_2	0.67	0.020	$[0.63, 0.71]$
	without X_2	0.65	0.020	$[0.61, 0.69]$
β_2	with X_2	-0.10	0.0.003	$[-0.11, -0.09]$

recurrent events of failure-repair on water network pipes

Global prediction of failures on the water network



recurrent events of failure-repair on water network pipes

individual prediction of failure on [U,V], at time T $(\leq U < V)$?

- at time T, use the adjusted model, the known history of the unit (age, covariate, number of past failures) to compute the Negative Binomial distribution for $N_i(V) - N_i(U)$.
- ranking of the sample by ordering $P[N_i(V) - N_i(U)]_{i=1\ldots n}$.
- use this ranking to provide a *preventive maintenance policy*.
- or do a graph of the Lift Curve to validate the prediction efficiency of the model.

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Conclusion - Bilan

Conclusion

- many models to handle dynamic intensity of recurrent events.
- the LEYP is one of them, with useful properties
- semiparametric framework has not been investigated (yet)
- recurrent events problems : epidemiology / biostatistics / reliability analysis : bridges exist.
- dynamic versus frailty models : a risk to badly identify dynamic components (in fact due to frailty component) : The negative Binomial distribution is also the marginal for Gamma mixed Poisson processes (Poisson with Gamma frailty)

Conclusion

- many models to handle dynamic intensity of recurrent events.
- the LEYP is one of them, with useful properties
- semiparametric framework has not been investigated (yet)
- recurrent events problems : epidemiology / biostatistics / reliability analysis : bridges exist.
- dynamic versus frailty models : a risk to badly identify dynamic components (in fact due to frailty component) : The negative Binomial distribution is also the marginal for Gamma mixed Poisson processes (Poisson with Gamma frailty)

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Thank you for your attention.